# On Electromagnetic Fields in a Periodically Inhomogeneous Chiral Medium

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Electromagnetic fields in a periodically inhomogeneous chiral medium are examined. The constitutive properties of the chiral medium vary along the $z$ axis, and reduced fields with prescribed $x$-variations are used. Coupled first-order differential equations are derived to describe the reduced fields. Two special cases – (i) piecewise constant inhomogeneity, and (ii) constant impedance inhomogeneity, – are discussed in detail, and solution procedures are provided.

## I. Introduction

Linearly polarized waves cannot exist in a homogeneous, isotropic chiral medium; but left- and right-circularly polarized (LCP and RCP) plane waves, with different phase velocities, are perfectly acceptable for this class of media [1, 2]. Chiral materials in nature exhibit the ramifications of their remarkable property only at and around optical frequencies, whence the term *natural optical activity* [3]. But this behaviour is due to the geometric handedness of the component molecules. Since optical activity is begotten by geometry, there is no reason why geometry at microscopic scales cannot be exploited to construct materials, the effects of whose chirality are observable at lower than optical frequencies.

Recently, advances have been made in constructing such artificially chiral media which betray the consequences of their microstructural chirality at frequencies in the lower GHz range [4]. There is also activity reported in synthesizing chiral macromolecules (e.g. [5, 6]), which could result in polymeric materials whose chirality may be observable at sub-optical frequencies. Coupled with the enhanced capabilities of manufacturing and characterizing thin films [7, 8, 9], these technological developments may lead to significant utilization of chiral materials in the near future for a variety of applications.

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These possibilities have recently spurred interest in the electromagnetic theory for chiral media. Recent developments have been summarized by us elsewhere [2], and a number of problems have been studied and solved [10–14]. With specific reference to layered media [15], in this paper we consider the problem of wave propagation in unidirectionally, periodically inhomogeneous chiral media, i.e., the constitutive properties are periodic functions of the $z$ coordinate. Coupled first order differential equations have been obtained to describe the electromagnetic field in the periodic medium. Two special cases have been discussed in detail. In what follows, boldface letters denote vector fields, small underlined letters denote column vectors, while capital underlined letters denote matrices.

## II. Preliminaries

An isotropic, unidirectionally inhomogeneous chiral medium is characterized by the constitutive equations

$$\boldsymbol{D}(\boldsymbol{r}) = \varepsilon(z)\,[\boldsymbol{E}(\boldsymbol{r}) + \beta(z)\nabla \times \boldsymbol{E}(\boldsymbol{r})],$$
$$\boldsymbol{B}(\boldsymbol{r}) = \mu(z)\,[\boldsymbol{H}(\boldsymbol{r}) + \beta(z)\nabla \times \boldsymbol{H}(\boldsymbol{r})], \tag{1a}$$

in which the permittivity $\varepsilon$, the permeability $\mu$, and the chirality parameter $\beta$ are all functions of $z$ alone. It is also assumed in the sequel that these constitutive parameters are all real; additionally, these parameters are assumed to possess periodic variations, i.e.,

$$\varepsilon(z+\Omega) = \varepsilon(z), \quad \mu(z+\Omega) = \mu(z), \quad \beta(z+\Omega) = \beta(z), \tag{1b}$$

where $\Omega$ is the period.

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Substitution of (1 a) into the Ampere-Maxwell and the Faraday-Maxwell equations leads to two first-order coupled differential equations, which are symbolically expressed in matrix notation as

$$\begin{bmatrix} \nabla \times \boldsymbol{E}(\boldsymbol{r}) \\ \nabla \times \boldsymbol{H}(\boldsymbol{r}) \end{bmatrix} = \underline{A}(z) \cdot \begin{bmatrix} \boldsymbol{E}(\boldsymbol{r}) \\ \boldsymbol{H}(\boldsymbol{r}) \end{bmatrix}. \tag{2a}$$

In deriving (2 a), as well as hereafter, an $\exp(-i\omega t)$ harmonic time-dependence has been assumed. The $2 \times 2$ matrix $\underline{A}(z)$ is periodic, i.e., $\underline{A}(z+\Omega)=\underline{A}(z)$; and it is given by

$$\underline{A}(z) = \gamma_1(z)\,\gamma_2(z) \cdot \begin{bmatrix} \beta(z) & i/\omega\,\varepsilon(z) \\ -i/\omega\,\mu(z) & \beta(z) \end{bmatrix}, \tag{2b}$$

in which the following definitions have been used:

$$k(z) = \omega\,[\varepsilon(z)\,\mu(z)]^{1/2}, \tag{3a}$$

$$\gamma_1(z) = k(z)/[1 - k(z)\,\beta(z)], \tag{3b}$$

$$\gamma_2(z) = k(z)/[1 + k(z)\,\beta(z)]. \tag{3c}$$

## III. Fourier Analysis and Reduced Fields

Without loss of generality, a harmonic field variation in the $xy$ plane can be assumed here; since the inhomogeneity is purely $z$-dependent, the fields can also be assumed independent of the $y$-direction. Thus, let

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{e}(z)\exp[i\varkappa x], \quad \boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{h}(z)\exp[i\varkappa x], \tag{5}$$

so that all further developments will be obtained for given $\{\omega, \varkappa\}$; $\boldsymbol{e}$ and $\boldsymbol{h}$ are called *reduced* fields. Parenthetically, it is mentioned that the much simpler treatment for $\varkappa = 0$ has been detailed by us elsewhere [16a].

Substitution of the Fourier decomposition (5) into the matrix differential equation (2 a) then yields the system equation

$$(\mathrm{d}/\mathrm{d}z)\,\underline{v}(z) = \underline{B}(z) \cdot \underline{v}(z). \tag{6a}$$

In this equation, the 4-vector $\underline{v}(z)$ is given by

$$\underline{v}(z) = \text{column}\,\{e_x(z); h_x(z); e_y(z); h_y(z)\}, \tag{6b}$$

and the $4 \times 4$ periodic matrix $\underline{B}(z)$ is given by

$$\underline{B}(z) = \begin{bmatrix} 0 & 0 & \beta(z)\,[\gamma_1(z)\,\gamma_2(z)+\varkappa^2] & i\,[\gamma_1(z)\,\gamma_2(z)-\varkappa^2]/\omega\,\varepsilon(z) \\ 0 & 0 & [\gamma_1(z)\,\gamma_2(z)-\varkappa^2]/i\,\omega\,\mu(z) & \beta(z)\,[\gamma_1(z)\,\gamma_2(z)+\varkappa^2] \\ -\beta(z)\cdot\gamma_1(z)\,\gamma_2(z) & \gamma_1(z)\,\gamma_2(z)/i\,\omega\,\varepsilon(z) & 0 & 0 \\ i\,\gamma_1(z)\,\gamma_2(z)/\omega\,\mu(z) & -\beta(z)\cdot\gamma_1(z)\,\gamma_2(z) & 0 & 0 \end{bmatrix}. \tag{6c}$$

Throughout this work it has been assumed that $|k(z)\,\beta(z)|<1$ for all $z$ to be considered. It should be mentioned that in a homogeneous chiral medium $\gamma_1$ and $\gamma_2$ are the wavenumbers of the LCP and the RCP fields, respectively, while $k$ is merely a shorthand notation.

The matrix $\underline{A}(z)$ is diagonalizable; i.e., it can be expressed in the form

$$\underline{A}(z) = \underline{T}(z) \cdot \underline{G}(z) \cdot \underline{T}^{-1}(z) \tag{4a}$$

with

$$\underline{G}(z) = \mathrm{diag}\,[-\gamma_2(z); \gamma_1(z)] \tag{4b}$$

being diagonal; the matrix $\underline{T}(z)$ is given by

$$\underline{T}(z) = \begin{bmatrix} 1 & 1 \\ i/\eta(z) & -i/\eta(z) \end{bmatrix}; \tag{4c}$$

$\underline{T}^{-1}(z)$ is the inverse of $\underline{T}(z)$ and

$$\eta(z) = [\mu(z)/\varepsilon(z)]^{1/2} \tag{4d}$$

carries the unit of an impedance. The diagonalizability of $\underline{A}(z)$ eliminates the necessity for the conversion of the system matrices in the sequel to their respective canonical Jordan forms [16].

This matrix $\underline{B}(z)$ too diagonalizes and has four eigenvalues: $\pm i\,[\gamma_1^2(z)-\varkappa^2]^{1/2}$ and $\pm i\,[\gamma_2^2(z)-\varkappa^2]^{1/2}$, which will be assumed distinct from each other for all $z$. The remaining components of the electromagnetic field can be obtained through the algebraic relations

$$-e_z(z) = [\varkappa/\omega\,\varepsilon(z)]\,h_y(z) + i\,\varkappa\,\beta(z)\,e_y(z), \tag{7a}$$

$$h_z(z) = [\varkappa/\omega\,\mu(z)]\,e_y(z) - i\,\varkappa\,\beta(z)\,h_y(z). \tag{7b}$$

In general, the solution of (6 a) cannot be obtained for arbitrary $\underline{B}(z)$, except numerically. Indeed, although the characteristics of such equations have been extensively studied, no general analytic solution procedure has been forthcoming. However, as per the Floquet-Lyapunov theorem [17–19], it can be stated that the solution of (6) must be of the form

$$\underline{v}(z) = \underline{F}(z) \cdot \exp\{\underline{K}\,z\} \cdot \underline{v}(0), \tag{8}$$

where $\underline{K}$ is a constant $4 \times 4$ matrix. The $4 \times 4$ matrix $\underline{F}(z)$ in (8) is periodic, i.e., $\underline{F}(z+\Omega)=\underline{F}(z)$; furthermore, $\underline{F}(0)=\underline{F}(\Omega)=\underline{I}$, the idempotent. Although $\underline{B}(z)$ is periodic, it must also be borne in mind that there is no guarantee that $\underline{v}(z)$ is also periodic.

For numerical computations, a more advantageous form may be obtained by a rearrangement of terms; to wit,

$$(\mathrm{d}/\mathrm{d}z)\cdot\begin{bmatrix} e_x+i\,e_y \\ h_x+i\,h_y \\ e_x-i\,e_y \\ h_x-i\,h_y \end{bmatrix} \tag{9}$$

$$=\gamma_2\,\gamma_2\begin{bmatrix} -i\beta & 1/\omega\varepsilon & 0 & 0 \\ -1/\omega\mu & -i\beta & 0 & 0 \\ 0 & 0 & i\beta & -1/\omega\varepsilon \\ 0 & 0 & 1/\omega\mu & i\beta \end{bmatrix}\begin{bmatrix} e_x+i\,e_y \\ h_x+i\,h_y \\ e_x-i\,e_y \\ h_x-i\,h_y \end{bmatrix}$$

$$+(\varkappa^2/2)\begin{bmatrix} -i\beta & -1/\omega\varepsilon & i\beta & 1/\omega\varepsilon \\ 1/\omega\mu & -i\beta & -1/\omega\mu & i\beta \\ -i\beta & -1/\omega\varepsilon & i\beta & 1/\omega\varepsilon \\ 1/\omega\mu & -i\beta & -1/\omega\mu & i\beta \end{bmatrix}\begin{bmatrix} e_x+i\,e_y \\ h_x+i\,h_y \\ e_x-i\,e_y \\ h_x-i\,h_y \end{bmatrix}$$

This equation may then be solved numerically, using the Runga-Kutta method [20] for instance. Perturbational solutions of (9) can also be obtained [21]. More importantly, however, the effect of obliqueness (i.e., $\varkappa$) has been isolated here. Consider the case when $\varkappa=0$: Then $\{e_x+i\,e_y, h_x+i\,h_y\}$ is independent of $\{e_x-i\,e_y, h_x-i\,h_y\}$, and the system (9) breaks up into two autonomous systems each of which involves the mixing of two eigenfields. But, when $\varkappa\neq 0$, these two systems are no longer autonomous so that (9) represents the coupling of four eigenfields.

There are two special cases, however, which can be discussed now, and which yield more readily than (6) or (9) to further analysis.

## IV. Piecewise Constant Case

Let the period $0\leqq z\leqq\Omega$ be broken up into $N$ layers, in each of which the medium properties are constant, as illustrated in Figure 1. In other words, let

$$\varepsilon(z)=\varepsilon_n,\quad \mu(z)=\mu_n,\quad \beta(z)=\beta_n; \qquad z_{n-1}\leqq z\leqq z_n;\quad n=1,2,\ldots,N \tag{10}$$

with $z_0=0$ and $z_N=\Omega$. Then, it is easy to see that the differential equation

$$(\mathrm{d}/\mathrm{d}z)\,\underline{v}(z)=\underline{B}_n\cdot\underline{v}(z),\quad z_{n-1}\leqq z\leqq z_n \tag{11a}$$

holds in the $n$-th layer. In this equation, the constant $4\times 4$ matrix $\underline{B}_n$ is given by

$$\underline{B}_n=\begin{bmatrix} 0 & 0 & \beta_n[\gamma_{1n}\gamma_{2n}+\varkappa^2] & i\,[\gamma_{1n}\gamma_{2n}-\varkappa^2]/\omega\varepsilon_n \\ 0 & 0 & [\gamma_{1n}\gamma_{2n}-\varkappa^2]/i\,\omega\mu_n & \beta_n[\gamma_{1n}\gamma_{2n}+\varkappa^2] \\ -\beta_n\gamma_{1n}\gamma_{2n} & \gamma_{1n}\gamma_{2n}/i\,\omega\varepsilon_n & 0 & 0 \\ i\,\gamma_{1n}\gamma_{2n}/\omega\mu_n & -\beta_n\gamma_{1n}\gamma_{2n} & 0 & 0 \end{bmatrix} \tag{11b}$$

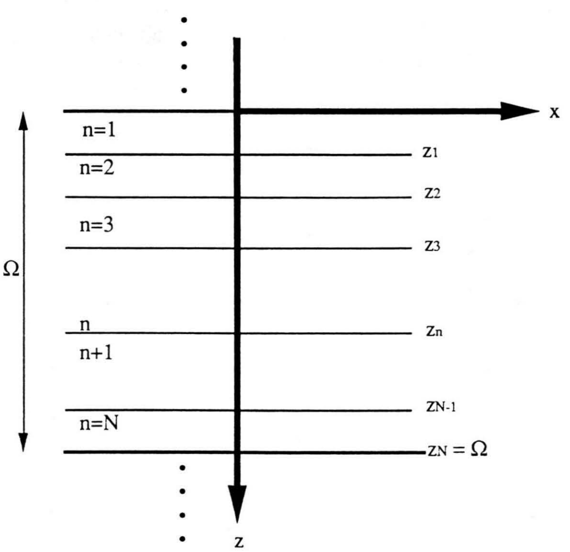


Fig. 1. To illustrate a piecewise constant, periodically inhomogeneous chiral medium.

with $\gamma_{1n}$ and $\gamma_{2n}$ defined as per (3) and (10).

Since $\underline{B}_n$ is a constant matrix, the solution of (11) is easily obtainable in the form [16]

$$\underline{v}(z)=\exp\{\underline{B}_n(z-z_{n-1})\}\cdot\underline{v}(z_{n-1}),\ z_{n-1}\leqq z\leqq z_n, \tag{12a}$$

whence

$$\underline{v}(z_n)=\exp\{\underline{B}_n(z_n-z_{n-1})\}\cdot\underline{v}(z_{n-1}), \tag{12b}$$

is an "output-input" relationship for the $n$-th slab. Repeated application of (12b) over the layer index $n$ gives rise to the "output-input" relationship

$$\underline{v}(\Omega)=\exp\{\underline{B}_N(\Omega-z_{N-1})\}\cdot\exp\{\underline{B}_{N-1}(z_{N-1}-z_{N-2})\}\cdot\ldots\cdot\exp\{\underline{B}_2(z_2-z_1)\}\cdot\exp\{\underline{B}_1 z_1\}\cdot\underline{v}(0). \tag{13a}$$

In general, the matrices $\underline{B}_n$ do not commute. However, if the layers are very thin, then the relation

$$\underline{v}(\Omega)=\exp\{\underline{M}\,\Omega\}\cdot\underline{v}(0) \tag{13b}$$

can be expected to hold with the $4 \times 4$ matrix $\underline{M}$ defined by the sum

$$\underline{M} = \sum_{n=1,2,\ldots,N} [\underline{B}_n \{z_n - z_{n-1}\}/\Omega]; \qquad (13\,c)$$

further, such an approximation may also be considered for slowly varying media. Comparison of (13 b) with the Floquet-Lyapunov solution (8), and the use of (12) then easily yields the matrices

$$\underline{K} = \underline{M}, \qquad (14\,a)$$

$$\underline{F}(z) = \exp\{\underline{B}_n(z - z_n\} \cdot \exp\left\{\sum_{m=1,2,\ldots,n} [\underline{B}_m(z_m - z_{m-1})]\right\} \cdot \exp\{-\underline{M}\, z\}, \qquad (14\,b)$$

$$z_{n-1} \leqq z \leqq z_n$$

which can be used in (8).

Provided both $\gamma_{1n}\Omega$ and $\gamma_{2n}\Omega$ are much smaller than unity for all $n$, the periodic matrix $\underline{F}(z)$ may be replaced by its average value $\underline{I}$. In this case, the electromagnetic field in the periodic, piecewise constant medium approximately follows the relationship

$$\underline{v}(z) = \exp\{\underline{M}\, z\} \cdot \underline{v}(0). \qquad (15\,a)$$

But $\underline{M}$ is diagonalizable. In other words, $\underline{M} = \underline{P} \cdot \underline{Q} \cdot \underline{P}^{-1}$, where $\underline{Q}$ is a diagonal matrix whose elements are the eigenvalues of $\underline{M}$, and the successive columns of $\underline{P}$ contain the corresponding eigenvectors of $\underline{M}$ [18]. Therefore, the field eigenstates in this approximation can be more elegantly expressed in the form

$$\underline{v}'(z) = \underline{P}^{-1} \cdot \underline{v}(z) \qquad (15\,b)$$

with

$$\underline{v}'(z) = \exp\{\underline{Q}\, z\} \cdot \underline{v}'(0). \qquad (15\,c)$$

## V. Constant Impedance Case

When the impedance $\eta(z)$ of the medium is constant although both $\mu(z)$ and $\varepsilon(z)$ are functions of $z$, the system (6) becomes considerably simpler. Let $\eta = [\mu(z)/\varepsilon(z)]^{1/2}$ be independent of $z$ and the sums $[e_x(z) \pm i\,\eta\, h_x(z)]$ and $[e_y(z) \pm i\,\eta\, h_y(z)]$ be formed. Also, let

$$\gamma_+(z) = \gamma_1(z), \quad \gamma_-(z) = -\gamma_2(z) \qquad (16)$$

for ease of notation. Then (6) breaks down into two autonomous systems, both of which are described by the equation

$$(\mathrm{d/d}z)\, \underline{v}_\pm(z) = \underline{B}_\pm(z) \cdot \underline{v}_\pm(z). \qquad (17\,a)$$

In (17 a), the 2-vector $\underline{v}_\pm(z)$ is specified as

$$\underline{v}_\pm(z) = \text{column}\,\{e_x(z) \pm i\,\eta\, h_x(z);\; e_y(z) \pm i\,\eta\, h_y(z)\}, \qquad (17\,b)$$

and the $2 \times 2$ matrix $\underline{B}_\pm(z)$ is given as

$$\underline{B}_\pm(z) = \begin{bmatrix} 0 & \gamma_\pm(z) - \varkappa^2/\gamma_\pm(z) \\ -\gamma_\pm(z) & 0 \end{bmatrix}. \qquad (17\,c)$$

Again, despite its relative simplicity, no general analytic solution procedure for (17) is available for arbitrary $\underline{B}_\pm(z)$, although it too may be examined numerically.

However, when $\underline{B}_\pm(z)$ is weakly dependent on $z$, perturbational solutions can be obtained rather easily. For the purpose of illustration, consider the system

$$(\mathrm{d/d}z)\, \underline{u}(z) = \underline{C}(z) \cdot \underline{u}(z), \qquad (18\,a)$$

$$\underline{C}(z) = \begin{bmatrix} 0 & \lambda(z) - \varkappa^2/\lambda(z) \\ -\lambda(z) & 0 \end{bmatrix}, \qquad (18\,b)$$

which is isomorphic with both of the two systems (17). With $\alpha$ as a small parameter, let a representation of the type

$$\underline{C}(z) \cong \underline{C}_0 + \alpha\, \underline{C}_1(z) + \alpha^2\, \underline{C}_2(z) + \text{higher order terms in } \alpha \qquad (19\,a)$$

be possible. Since $\underline{C}(z)$ is periodic, $\underline{C}_n(z)$ also are, and Fourier expansions of the type

$$\underline{C}_n(z) = \sum_{m=\pm 1, \pm 2, \ldots} \exp(2\pi\, i m z/\Omega)\, \underline{C}_{nm}, \quad n > 0 \qquad (19\,b)$$

can be made in which $\underline{C}_{nm}$ are constant matrices.

Let the $2 \times 2$ matrix $\underline{U}(z)$ contain the fundamental solutions of (18) as its column vectors; this matrix $\underline{U}(z)$ is called the matrizant [22], and the solutions of the system (18) can be constructed from linear combinations of the column vectors of the matrizant. As per the Floquet-Lyapunov theorem,

$$\underline{U}(z) = \underline{F}(z) \exp(\underline{K}\, z), \qquad (20)$$

*vide* (8). Correct to the second order in $\alpha$, perturbational analysis then gives [19] the following results:

$$\underline{F}(z) \cong \underline{I} + \alpha\, \underline{F}_1(z) + \alpha^2\, \underline{F}_2(z), \qquad (21\,a)$$

$$K \cong K_0 + \alpha\, \underline{K}_1 + \alpha^2\, \underline{K}_2. \qquad (21\,b)$$

The computation of the various matrices involved in (21 a, b) takes place as follows:

$$\underline{K}_0 = \underline{C}_0, \qquad (22\,a)$$

$$\underline{K}_1 = \left[(1/\Omega) \int_0^\Omega \mathrm{d}\tau\, \underline{C}_1(\tau)\right], \qquad (22\,b)$$

$$\underline{K}_2 = \left[(1/\Omega) \int_0^\Omega \mathrm{d}\tau\, \underline{C}_2(\tau)\right] + \sum_{m \neq 0} \underline{C}_{1,-m} \cdot \Re_m\{\underline{C}_{1,m}\}, \qquad (22\,c)$$

$$\underline{\Phi}_m = \underline{C}_{2,m} - \mathfrak{R}_m\{\underline{C}_{1,m}\} \cdot \underline{K}_1 + \sum_{j+k=m} \underline{C}_{1,k} \cdot \mathfrak{R}_j\{\underline{C}_{1,j}\}, \tag{22d}$$

$$\underline{F}_1(z) = \sum_{m \neq 0} \exp(2\pi i m z/\Omega)\, \mathfrak{R}_m\{\underline{C}_{1,m}\}, \tag{22e}$$

$$\underline{F}_2(z) = \sum_{m \neq 0} \exp(2\pi i m z/\Omega)\, \mathfrak{R}_m\{\underline{\Phi}_m\}. \tag{22f}$$

Here, the matrix function $\mathfrak{R}_m\{\underline{W}\}$ is the solution of the matrix equation

$$\underline{K}_0 \cdot \mathfrak{R}_m\{\underline{W}\} - \mathfrak{R}_m\{\underline{W}\} \cdot \underline{K}_0 + \underline{W} = (2\pi i m/\Omega)\, \mathfrak{R}_m\{\underline{W}\}. \tag{22g}$$

For the perturbational solution to hold, however, the two eigenvalues of the constant matrix $\underline{C}_0$ must be incongruent modulo $2\pi i/\Omega$.

As an example of this perturbational procedure, let us consider the case of the periodic inhomogeneity specified as

$$\lambda(z) = \lambda_s[1 + \alpha \cos(2\pi z/\Omega)] \tag{23}$$

subject to the restriction $|\alpha| \ll 1$. On applying the aforementioned procedure, it is easy to see that, correct to first order in $\alpha$,

$$\underline{C}_0 = \begin{bmatrix} 0 & \lambda_s - \varkappa^2/\lambda_s \\ -\lambda_s & 0 \end{bmatrix},$$

$$[\underline{C}_1] = \begin{bmatrix} 0 & \lambda_s + \varkappa^2/\lambda_s \\ -\lambda_s & 0 \end{bmatrix} \cos\varphi(z), \tag{24}$$

while $\underline{C}_n$, $n > 1$, are null matrices. Using the notation

$$\varphi(z) = 2\pi z/\Omega = 2pz, \quad p = \pi/\Omega, \tag{25}$$

the perturbation procedure gives the following results to determine the matrizant $\underline{U}(z)$ as per (20)–(22):

$$\underline{K}_0 = \begin{bmatrix} 0 & \lambda_s - \varkappa^2/\lambda_s \\ -\lambda^s & 0 \end{bmatrix}, \tag{26a}$$

$$\underline{K}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{26b}$$

$$\underline{K}_2 = (\varkappa/2)^2 [\varkappa^2 + p^2 - \lambda_s^2]^{-1} \begin{bmatrix} 0 & \lambda_s + \varkappa^2/\lambda_s \\ \lambda_s & 0 \end{bmatrix}, \tag{26c}$$

$$2\lambda_s p[\varkappa^2 + p^2 - \lambda_s^2]\, \underline{F}_1 = \begin{bmatrix} -\varkappa^2 \lambda_s p \cos\varphi & \{\varkappa^2[\lambda_s^2 + p^2] - \lambda_s^2[\lambda_s^2 - p^2]\}\sin\varphi \\ \lambda_s^2[\lambda_s^2 - p^2]\sin\varphi & \varkappa^2\lambda_s p\cos\varphi \end{bmatrix}. \tag{26d}$$

In this solution, the matrix $\underline{F}(z)$ has been estimated to the first order in $\alpha$, but the matrix $\underline{K}$ has been obtained correct in the second order since $\underline{K}_1$ is identically zero. It should be noted that the eigenvalues of $\underline{K}$ are

$$\pm[\zeta - 1]^{1/2}([\zeta - 1]\varkappa^2 + [\zeta + 1]\lambda_s^2)^{1/2}, \tag{27a}$$

where

$$\zeta = \alpha^2 \varkappa^2/4\,[\varkappa^2 + p^2 - \lambda_s^2]. \tag{27b}$$

Also, the two eigenvalues of $\underline{C}_0$ are $\pm i[\lambda_s^2 - \varkappa^2]^{1/2}$; therefore, the solution (26) blows up when $\varkappa^2 + p^2 = \lambda_s^2$. Finally, as a check, let $\alpha = 0$ in (23). Then $\underline{F}(z) = \underline{I}$, $\underline{K} = \underline{C}_0$, $\zeta = 0$, which conform to the results obtainable for a homogeneous chiral medium [1].

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